A *SCHOLIUM* ON THE INTEGRAL OF $\sin(x)/x$ AND RELATED TOPICS

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Abstract. Leisurely coaching is given about $\sin(x)/x$, its integral, and its many relations to Laplace transforms, characteristic functions, asymptotic expansions, special integrals, product formulas, and inversion formulas. The document is still evolving, and it is ragged in parts. Someday some parts may be used in some book, but don't hold your breath.

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1. The Laplace Transform of Sin(x)

One of the great workhorses of mathematics is the Laplace transform of the function $x \mapsto \sin(x)$:

$$\phi(\lambda) = \int_0^\infty \sin(x) e^{-\lambda x}\, dx = \frac{1}{1+\lambda^2}. \tag{1}$$

This integral pops up everywhere, and it has an endearing number of derivations. We will look at five of them before putting the formula (1) to work on other problems.

First Proof: Series Expansion

This method takes more ink than some derivations, but it has the benefit of being completely straightforward. The raw ingredients are just the Taylor series expansion of $\sin x$ and the easy integral

$$\int_0^\infty x^n e^{-\lambda x}\, dx = \frac{n!}{\lambda^{n+1}}. \tag{2}$$

This integral can be obtained by integration by parts, but interchange of differentiation and integration gives a somewhat nicer derivation. Beginning with trivial integral

$$\int_0^\infty e^{-\lambda x}\, dx = \frac{1}{\lambda},$$

one just differentiates n times to get (2). Alternatively, one can view (2) as just a special case of the definition and valuation of the Gamma function, but, at the beginning at least, we stick to first principles. The Gamma function will have its turn later.

Now, to get the Laplace transform of $\sin x$, we expand $\sin x$ in the Taylor series about $x = 0$ and integrate term by term to find

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$$
\begin{aligned}
\int_0^\infty \sin(x)e^{-\lambda x}\,dx &= \int_0^\infty \sum_{n=0}^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!}e^{-\lambda x}\,dx \\
&= \sum_{n=0}^\infty \frac{(-1)^k}{(2k+1)!}\int_0^\infty x^{2k+1}e^{-\lambda x}\,dx \\
&= \sum_{n=0}^\infty \frac{(-1)^k}{\lambda^{2k+2}} = \frac{1}{\lambda^2}\sum_{n=0}^\infty \frac{(-1)^k}{\lambda^{2k}} = \frac{1}{1+\lambda^2},
\end{aligned}
$$

where the last step uses summation of the geometric series.

This derivation follows the famous the *analysis-synthesis* paradigm where one takes something apart, manipulates the parts, and reassembles the parts at the end. Given the analysis step introduces some additional complexity, it is remarkable how often one manages to regain simplicity at the end.

Second Proof: Integration of Euler's Formula

There is another direct path to the Laplace transform of $\sin x$ that has the added benefit of killing two Laplace transform birds with one stone. We bundle $\sin x$ and $\cos x$ together through Euler's formula $e^{ix} = \cos(x) + i\sin(x)$, so integration gives us

$$\int_0^\infty e^{ix}e^{-\lambda x}\,dx = \int_0^\infty e^{-x(\lambda - i)}\,dx = \frac{1}{\lambda - i} = \frac{\lambda + i}{\lambda^2+1}.$$

This integral now tells us everything we want to know; specifically, taking the real and imaginary parts gives us

$$\int_0^\infty \cos(x)e^{-\lambda x}\,dx = \frac{\lambda}{1+\lambda^2} \quad\text{and}\quad \int_0^\infty \sin(x)e^{-\lambda x}\,dx = \frac{1}{1+\lambda^2}. \tag{3}$$

This derivation is certainly slick, and it may offer the easiest way to remember the formulas for the Laplace transforms of $\sin x$ and $\cos x$. Still, to be completely fair, something more needs to be said about the formula

$$\int_0^\infty e^{-xz}\,dx = \frac{1}{z} \quad \text{when } \operatorname{Re} z > 0. \tag{4}$$

Given some experience with analytic functions, this is not mysterious, but neither should it be regarded as completely obvious or trivial. A formal confirmation of this identity takes several steps and uses some machinery.

First one checks that the integral on the left determines an analytic function for all $\operatorname{Re} z > 0$; there are several ways to do this. Second, one notes that the function on the right side is also analytic in this domain. Finally, we know from real variable calculus that the formula is valid for real positive real values of z. Thus, we find ourselves with two analytic functions that agree on a set that contains a limit point. It is a basic theorem of complex analysis, any two such analytic functions must agree on their common domain of definition. In our case, this says that the formula (4) must hold for all $\operatorname{Re} z > 0$.

Third Proof: Integration by Parts

The third approach to the Laplace transform of $\sin x$ goes via integration by parts. With just a rough mental calculation, it reasonably clear that after integrating by parts two times one must come to an equation that can be solved for $\phi(\lambda)$.

It is not always pleasant to track of the boundary values and the minus signs, but one can soldier along:

$$\begin{aligned}\phi(\lambda) &= \int_0^\infty \sin(x)e^{-\lambda x}\,dx = \int_0^\infty \sin(x)(-\lambda^{-1}e^{-\lambda x})'\,dx \\ &= \lambda^{-1}\int_0^\infty \cos(x)e^{-\lambda x}\,dx = \lambda^{-1}\int_0^\infty \cos(x)(-\lambda^{-1}e^{-\lambda x})'\,dx \\ &= \lambda^{-2}\left\{-\Big|_0^\infty \cos(x)e^{-\lambda x} - \int_0^\infty \sin(x)e^{-\lambda x}\,dx\right\} = \lambda^{-2}\{1-\phi(\lambda)\}.\end{aligned}$$

When we solve for $\phi(\lambda)$ we recover our familiar formula for the Laplace transform of $\sin x$ for a third time.

Fourth Proof: Derivation via the ODE for Sin

This time we start with the characterization of $\sin x$ by the ordinary differential equation

$$f''(x) + f(x) = 0 \qquad f(0) = 0,\ f'(0) = 1,$$

and we ask, "What do we get if we just take the Laplace transform $\mathcal{L}$ of this equation?"

All we need for the calculation are the general formulas for the transforms of derivatives:

$$\text{(5)}\quad \mathcal{L}[f'](\lambda) = -f(0) + \lambda\mathcal{L}[f](\lambda) \quad \text{and} \quad \mathcal{L}[f''](\lambda) = -f'(0) - \lambda f(0) + \lambda^2\mathcal{L}[f](\lambda).$$

When we apply the second rule to the ODE for $\sin x$, we get the equation

$$-1 + \lambda^2\mathcal{L}[f](\lambda) + \mathcal{L}[f](\lambda) = 0,$$

which we solve to find for the last time that $\phi(\lambda) = \mathcal{L}[\sin](\lambda) = (1+\lambda^2)^{-1}$.

If the rules (5) for the Laplace transforms of derivatives are not already deeply familiar to you, there are some healthy tests to confirm that they are properly remembered. The first rule is easy enough to rederive in ones head, but it is also nice to note that it is easily checked by taking $f(x) = 1$. Similarly, one can test the second rule with $f(x) = x$, but it is only a little more demanding to do the mental test with $f(x) = \sin x$ and $f(x) = \cos x$, assuming by this time that Laplace transforms of these functions are a reliable part of ones knowledge.

The ODE derivation is closely connected with the integration by parts method. After all, the rules for the transforms of derivatives are special cases of the integration by parts formula. Thus, the apparent simplicity of the ODE argument is partially psychological; knowledge of the transformation rules (5) serve to compartmentalize and modularize an ancestral integration by parts argument that lurks just below the surface.

Exercise. Something that physicist always remember (and mathematicians sometimes forget) is that frequency and phase enrich the meaning and interpretation of the trigonometric functions. This lesson holds as well for their Laplace transforms. Use what we have found to show that one has the beautiful symmetric pair:

$$\sin(\omega x + \phi) \mapsto \frac{\lambda\sin\phi + \omega\cos\phi}{\lambda^2+\omega^2} \quad \text{and} \quad \cos(\omega x + \phi) \mapsto \frac{\lambda\cos\phi - \omega\sin\phi}{\lambda^2+\omega^2}.$$

Exercise. Show that one has

$$\int_0^\infty \sin^2(x) e^{-\lambda x}\, dx = \frac{2}{\lambda(4+\lambda^2)}.$$

Hint: This might look unappetizing at first glance, but when you recall that $\sin^2(x)$ has the nice derivative $2\sin(x)\cos(x) = \sin(2x)$ things look rosy. If we know the Laplace transform of f', we can easily get the Laplace transform of f. The formula also has a built-in check; since $\sin^2(x) \sim x^2$ for $x \to \infty$ we can anticipate that the Laplace transform needs to behave like $2/\lambda^3$ as $\lambda \to \infty$.

Fifth Proof: Derivation via the Matrix Exponential

We have the famous pair

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad e^{tA} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

and we also have the matrix integral

$$\int_0^\infty e^{-\lambda t} e^{tA}\, dt = \int_0^\infty e^{-t(\lambda - A)}\, dt = (\lambda - A)^{-1}. \tag{6}$$

Moreover, we can find the inverse explicitly,

$$(\lambda - A)^{-1} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}^{-1} = \frac{1}{1+\lambda^2} \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}.$$

Now, if we simply do the first integral of (6) in then coordinate by coordinate fashion, then we can equate the results to find

$$\begin{pmatrix} \mathcal{L}[\cos](\lambda) & \mathcal{L}[\sin](\lambda) \\ -\mathcal{L}[\sin](\lambda) & \mathcal{L}[\cos](\lambda) \end{pmatrix} = \frac{1}{1+\lambda^2} \begin{pmatrix} \lambda & 1 \\ -1 & \lambda \end{pmatrix}.$$

One can say that this argument is just a typographical variation on the second proof where we used Euler's formula. That is true, but it misses some distinctions. Here the factor $1+\lambda^2$ has the interesting interpretation as a determinant, where earlier it showed up when we found the real and imaginary parts of a ratio of complex numbers. Also, the proof via Euler's formula leaves one without any obvious routes to further generalization, but the matrix exponential method suggests many new possibilities.

2. Introducting the $1/x$ Factor

We can now easy find the Laplace transform of our favorite function. In particular, if we start with the Laplace transform for $\sin(x)$, one does not need long to discover the exquisite formula,

$$\int_0^\infty \frac{\sin x}{x} e^{-\lambda x}\, dx = \frac{\pi}{2} - \arctan(\lambda). \tag{7}$$

Perhaps the most pleasing path to the discovery of (7) goes through Fubini's theorem and the representation

$$\frac{1}{x} = \int_0^\infty e^{-xy}\, dy.$$

Specifically, we have

$$\int_0^\infty \frac{\sin x}{x} e^{-\lambda x}\,dx = \int_0^\infty \left[\int_0^\infty e^{-xy}\,dy\right](\sin x)e^{-\lambda x}\,dx$$
$$= \int_0^\infty \left[\int_0^\infty (\sin x)e^{-x(\lambda+y)}\,dx\right]dy = \int_0^\infty \frac{1}{1+(\lambda+y)^2}\,dy,$$

where in the last step we used our formula (1) for the Laplace transform of $\sin(x)$. Now if we set $u = y + \lambda$ we have

$$\int_0^\infty \frac{1}{1+(\lambda+y)^2}\,dy = \int_\lambda^\infty \frac{1}{1+u^2}\,du = \frac{\pi}{2} - \arctan(\lambda).$$

Naturally, there are several other ways to obtain the integral (7).

Exercise. Prove the identity (7) by the series expansion method. The calculations are very close to those we used in our first derivation of the Laplace transform for $\sin(x)$. This time one just needs to recognize the Taylor expansion of $\arctan(x)$.

Limit at Zero of the Laplace Transform

Since $\arctan(0) = 0$, it is natural to consider the identity (7) as $\lambda \to 0$ and to infer that one has

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \tag{8}$$

This formula is true, but, since $(\sin x)/x$ is not absolutely integrable on $[0, \infty)$, a few extra steps are needed to pass honestly from (7) to (8). What we do get immediately from (7) is that for any $A \geq 0$ one has

$$\int_0^A \frac{\sin x}{x} e^{-\lambda x}\,dx - \frac{\pi}{2} = \int_A^\infty \frac{\sin x}{x} e^{-\lambda x}\,dx - \arctan(\lambda). \tag{9}$$

Now, to go from (9) to (8), we just need some tools from to help us estimate the second integral; these are developed in the next section.

3. Second Mean Value Theorem for Integrals

If $D : [a, b] \to \mathbb{R}$ is a non-increasing function, then for any integrable function $f : [a, b] \to \mathbb{R}$, there is a $\xi \in [a, b]$ such that

$$\int_a^b f(x)D(x)\,dx = D(a)\int_a^\xi f(x)\,dx + D(b)\int_\xi^b f(x)\,dx. \tag{10}$$

This is known as the *second mean value theorem of integral calculus*, though it would be nice to have a more charismatic name. It has sometimes been called "the most useful fact that is no longer covered in calculus classes."

Both sides of the relation (10) are linear functions of D, so if the identity holds for some D then it also holds for $\alpha D + \beta$ for any constants α and β. Consequently, to prove (10) one can assume without loss of generality that $D(a) = 1$ and $D(b) = 0$. Assuming these relations, it then suffices to prove that there is a value $\xi \in [a, b]$ such that

$$\int_a^b f(x)D(x)\,dx = \int_a^\xi f(x)\,dx. \tag{11}$$

If we now set

$$F(x) = \int_a^x f(x)\,dx,$$

then from $F(a) = 0$ and $D(b) = 0$ we get by integration by parts that

$$\int_a^b f(x)D(x)\,dx = \int_a^b F(x)\{-D'(x)\}\,dx. \tag{12}$$

The function $x \mapsto -D'(x)$ is non-negative and its integral over $[a, b]$ is equal to one, so we have

$$\min\{F(x) : a \le x \le b\} \le \int_a^b F(x)\{-D'(x)\}\,dx \le \max\{F(x) : a \le x \le b\}.$$

Since F is continuous, F takes on all values between its maximum and minimum so there is a $\xi \in [a, b]$ such that

$$\int_a^b F(x)\{-D'(x)\}\,dx = F(\xi).$$

This observation and (12) then give us the conjectured identity (11).

4. Existence of the Integral

The much loved integral

$$I = \int_0^\infty \frac{\sin x}{x}\,dx \tag{13}$$

is not absolutely convergent, so to give it meaning we define it to be the value of the limit

$$I = \lim_{M\to\infty} \int_0^M \frac{\sin x}{x}\,dx.$$

One can use *ad hoc* estimates to show that this limit exists, but it is useful to know how to deal systematically with such problems.

Here if one takes the decreasing function $D(x) = 1/x$, then the second mean value for integrals (10) tells us that for any $0 \le A \le B < \infty$ that there is a $\xi \in [A, B]$ such that

$$\int_A^B \frac{\sin x}{x}\,dx = \frac{1}{A}\int_A^\xi \sin x\,dx + \frac{1}{B}\int_\xi^B \sin x\,dx.$$

Direct integration tells us that for any α and β we have

$$-2 \le \int_\alpha^\beta \sin x\,dx \le 2.$$

This gives us the classic inequality

$$\left|\int_A^B \frac{\sin x}{x}\,dx\right| \le \frac{2}{A} + \frac{2}{B} \le \frac{4}{A}, \tag{14}$$

and nothing more is needed to prove the existence of the integral (13).

Still, there is more to do. By the formula (9) and the second mean value theorem we also have

$$\left|\int_0^A \frac{\sin x}{x} e^{-\lambda x}\,dx - \frac{\pi}{2}\right| \le \frac{2e^{-\lambda A}}{A} + |\arctan(\lambda)|.$$

Now if we let $\lambda \to 0$ then

$$\left| \int_0^A \frac{\sin x}{x}\, dx - \frac{\pi}{2} \right| \le \frac{2}{A}, \tag{15}$$

which is a useful complement to our earlier estimate (14).

Either of the bounds (14) and (15) are enough to prove the validity of the identity (8). These relations actually refine (8); they have all of the information in (8) and more.

An Alternative Passage — Integration by Parts

The second mean value theorem is extremely useful, but it is only fair to admit that the existence of the integral (13) can be obtained directly by integration by parts. Specifically, for all $a > 0$ we have

$$\int_0^a \frac{\sin t}{t}\, dt = \int_0^a (1-\cos t)' \frac{1}{t}\, dt = \frac{1-\cos a}{a} + \int_0^a \frac{1-\cos t}{t^2}\, dt. \tag{16}$$

The last integral converges absolutely, so letting $a \to \infty$ shows the limit of the first integral also exists. This representation has many uses and it should never be far out of mind. A major technical benefit of the second integral is that the integrand is nonnegative, and this often makes light work of estimations.

5. One More Term

There is an way to refine the estimate (15) that works for many related integrals. The idea is to apply integration by parts to the remainder term. We first write

$$\int_0^t \frac{\sin x}{x}\, dx = \frac{\pi}{2} - \int_t^\infty \frac{\sin x}{x}\, dx,$$

then we observe that

$$\begin{aligned}\int_t^\infty \frac{\sin x}{x}\, dx &= -\int_t^\infty \frac{(\cos x)'}{x}\, dx = -\Big|_t^\infty \frac{\cos x}{x} - \int_t^\infty \frac{\cos x}{x^2}\, dx \\ &= \frac{\cos t}{t} - \int_t^\infty \frac{\cos x}{x^2}\, dx.\end{aligned}$$

When we use the second mean value theorem to estimate the last integral, we find

$$\left| \int_0^t \frac{\sin x}{x}\, dx - \frac{\pi}{2} + \frac{\cos t}{t} \right| \le \frac{2}{t^2}. \tag{17}$$

Naturally, for an even better approximation, one could do another integration by parts before estimating the remaining integrals.

Exercise. Use the suggested method to show that one has

$$\int_0^t \frac{\sin x}{x}\, dx = \frac{\pi}{2} - \frac{\cos t}{t} + \frac{\sin t}{t^2} + O(1/t^3).$$

In the same way one can develop an approximation of order $O(1/t^n)$ for any $n \ge 1$.

Exercise. Show that one has

$$\int_0^\infty \sin^2(x) e^{-\lambda x}\, dx = \frac{2}{\lambda(4+\lambda^2)}. \tag{18}$$

Hint: There is method to this madness. The key observation is that $\sin^2(x)$ has a nice derivative. Specifically, $(\sin^2(x))' = 2\sin x \cos x = \sin 2x$ and we know the Laplace transform of $\sin 2x$. The method? Any time you know the Laplace transform of f' it is trivial to find the Laplace transform of f.

Exercise. Show that "under normal circumstances" if one has $f(x) \sim ax^n$ as $x \to 0$ then one also has

$$\int_0^\infty f(x)e^{-\lambda x}\,dx \sim \frac{an!}{\lambda^{n+1}} \quad \text{as } \lambda \to \infty.$$

Since $\sin^2(x) \sim x^2$ as $x \to 0$, we expect its Laplace transform to behave like $2/\lambda^3$ as $\lambda \to \infty$. This gives us a check on the computation (18).

6. Partial Laplace Transform

Integration by parts gives you a new way to look at an integral, no information is lost by the transformation, and, unlike series expansion, the complexity is only modestly increased. Here, when we use integration by parts to compute the *partial* Laplace transform of $\sin x$ we find a truly notable formula

$$\int_0^a \sin(x)e^{-\lambda x}\,dx = \frac{1-(\lambda \sin a + \cos a)e^{-\lambda a}}{1+\lambda^2}. \tag{19}$$

A partial Laplace transform is always more complicated than the full Laplace transform; at a minimum one has to accommodate an extra parameter for the upper limit. Still, the extra parameter can be a benefit. Moreover, it often nicer to deal with an integral that has compact support.

Here, if integration by parts does not appeal to you, Euler's formula provides an alternative derivation of (19). Specifically, one can take the imaginary part of the elementary complex integral

$$\int_0^a e^{ix}e^{-\lambda x}\,dx = \frac{i+\lambda}{1+\lambda^2}\{1-e^{a(i-\lambda)}\}.$$

Partial Laplace transforms have many technical benefits, and often they can be used to make rigorous a heuristic calculation — one where formal manipulations conditionally convergent integrals have been used. The partial Laplace transform is also a useful tool in its own right, as the next representation suggests.

The $1/x$ Factor and Another Integral Representation

The partial Laplace transform (19) leads us easily to an an important formula for the integral of $(\sin x)/x$ over $[0,a]$. To see this, we just modify our earlier calculations. Specifically, we consider the integrand $e^{-xy}\sin(x)$ on the domain $[0,a]\times[0,\infty)$ and we use Fubini's theorem to equate two evaluations of the double integral. By doing the integral first on y we get our traditional factor $1/x$, and by doing the integration first over x brings in the partial Laplace transform (19). When equate the two ways of doing the double integral we find

$$\int_0^a \frac{\sin(x)}{x}\,dx = \frac{\pi}{2} - \cos a \int_0^\infty \frac{e^{-ay}}{1+y^2}\,dy - \sin a \int_0^\infty \frac{ye^{-ay}}{1+y^2}\,dy. \tag{20}$$

This representation can often be used to refine what one might learn from the direct use of the second intermediate value theorem.

For example, from the trivial bound $1/(1+y^2) \le 1$ one finds from (20) that

$$\left| \int_0^a \frac{\sin(x)}{x}\, dx - \frac{\pi}{2} \right| \le \frac{|\cos a|}{a} + \frac{|\sin a|}{a^2}, \tag{21}$$

which is a thin but curious refinement of the bound (15) that we found from the second mean value theorem for integrals. Also, in either (17) or (21), the special choice $a = (2k+1)\pi/2$ leads to a quadratic error bound,

$$\left| \int_0^{(2k+1)\pi/2} \frac{\sin(x)}{x}\, dx - \frac{\pi}{2} \right| \le \frac{4}{\pi^2(2k+1)^2}.$$

This special, of course, but still it may be better than one might have expected to be possible.

Exercise. Check the formula (20) by direct differentiation. The inevitable cancellations are strangely enjoyable; try it!

Exercise. Cauchy-Schwarz gives us $|u \sin a + \cos a| \le \sqrt{1+u^2}$. Now argue from (20) that for $a > 0$ one has

$$\left| \int_0^a \frac{\sin(x)}{x}\, dx - \frac{\pi}{2} \right| < \frac{1}{a}.$$

7. Frullani or Not So Frullani

For a large class of functions $f : \mathbb{R}^+ \to \mathbb{R}$, one has the Frullani identity

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = [f(0) - f(\infty)] \log(\frac{b}{a}), \quad \text{for } 0 \le a, 0 \le b, \tag{22}$$

where $f(\infty)$ is shorthand for the limit of $f(x)$ as $x \to \infty$. The classic example is Frullani's exponential integral,

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log(\frac{b}{a}), \quad \text{for } 0 \le a, 0 \le b. \tag{23}$$

One can prove in several ways, including the Fubini method of Section 2.

There is a general *direct method* that one can use to identities like (22). First one fixes a small $\epsilon > 0$ and a large $0 < M < \infty$. One then separates the integrals, changes variables, and invokes an intermediate value theorem to get

$$\begin{aligned} \int_\epsilon^M \frac{f(ax) - f(bx)}{x}\, dx &= \int_{a\epsilon}^{b\epsilon} \frac{f(x)}{x}\, dx - \int_{aM}^{bM} \frac{f(x)}{x}\, dx \\ &= f(x_\epsilon) \int_{a\epsilon}^{b\epsilon} \frac{1}{x}\, dx - f(x_M) \int_{aM}^{bM} \frac{1}{x}\, dx \\ &= \{f(x_\epsilon) - f(x_M)\} \log(\frac{b}{a}), \end{aligned}$$

where the values $x_\epsilon \in [a\epsilon, b\epsilon]$ and $x_M \in [aM, bM]$ are given by the first mean value theorem for integrals. In many circumstances the desired Frullani identity now follows when we let $\epsilon \to 0$ and $M \to \infty$. This works flawlessly when $f(x) = e^{-x}$.

For the function $f(x) = \sin x$ the Frullani identity (22) is more nuanced. Of course, we do have the (relatively uninteresting) integral

$$\int_0^\infty \frac{\sin ax - \sin bx}{x}\, dx = 0 \quad \text{for } 0 \le a, 0 \le b,$$

and this has the Frullani form if we interpret $\sin(\infty)$ to be zero. One could argue about how to justify that interpretation, and there are other integrals for which a similar issue emerges. Ramanujan already observed in Notebook I (cf. Albano et al. (2010)) that one has the more interesting integral

$$\int_0^\infty \frac{\cos ax - \cos bx}{x}\, dx = \log(\frac{b}{a}), \quad \text{for } 0 \le a, 0 \le b, \tag{24}$$

and, again, this has the Frullani form provided that we interpret $\cos(\infty)$ to be zero.

Still, there is nothing mysterious about (24), and one can prove it by first noting that the direct method is good enough to show that for $\lambda > 0$ that we have

$$\int_0^\infty \frac{e^{-\lambda ax}\cos ax - e^{-\lambda bx}\cos bx}{x}\, dx = \log(\frac{b}{a}), \quad \text{for } 0 \le a, 0 \le b. \tag{25}$$

Now one lets $\lambda \to 0$ and argues as in Section 4 that the limit really is given by the conditionally convergent integral (24).

Introduction of a convergence factor like $e^{-\lambda x}$ is a common trick. It greatly extends the applicability of the direct method for proving Frullani identities, and the only price one has to pay is an additional limit argument. Moreover, this argument is often routine with help from the second mean value for integrals.

Exercise. Complete the argument that (25) implies (24). You will probably want to make use of the second mean value theorem for integrals.

Exercise. Show by the direct method that one has for $a > 0, b > 0$ that

$$\int_0^\infty \left\{\frac{\sin ax}{ax} - \frac{\sin bx}{bx}\right\} \frac{dx}{x} = \log(\frac{b}{a}).$$

Exercise. Show that Frullani's exponential integral (23) does indeed follow from the Fubini method of Section 2.

8. The Curious Discovery of an Integral Shift

If we consider the inhomogeneous equation

$$y''(t) + y(t) = \frac{1}{t} \quad \text{when } t > 0, \tag{26}$$

there are several methods for finding a solution. If we look for a solution that has the general form of a Laplace transform

$$y(t) = \int_0^\infty h(x)e^{-tx}\, dx,$$

then we see that we need

$$\int_0^\infty (1+x^2)h(x)e^{-tx}\, dx = \int_0^\infty e^{-tx}\, dx.$$

Hence, if we take $h(x) = 1/(1+x^2)$, then we see that one solution of (26) is given by the integral

$$I_1(t) = \int_0^\infty \frac{e^{-tx}}{1+x^2}\, dx.$$

This is a specialized method, but it is certainly worth trying whenever the inhomogeneous term is itself a Laplace transform. Without this condition, the method obviously cannot work.

The more systematic approach to an inhomogeneous equation (26) is to use the method of Green's functions, or the method of variations of parameters. This method works quite smoothly, and the Green's function solution of (26) is given by another integral

$$I_2(t) = \int_t^\infty \frac{\sin(x-t)}{x}\,dx.$$

Of course, once these integrals are written down, one can verify by differentiation that they are indeed solutions of (26), but are they different? It turns out that they are not.

To see this, first note that difference $I_1(t)-I_2(t)$ solves the homogenous equation $y''(t)+y(t)=0$, and all solutions of this homogenous equation are periodic with period 2π. The functions $I_1(x)$ and $I_2(x)$ both go to zero as $x \to \infty$, so, for $I_1(t)-I_2(t)$ to be periodic, it must be zero for all $t>0$. In other words, we have the curious identity,

$$\int_t^\infty \frac{\sin(x-t)}{x}\,dx = \int_0^\infty \frac{e^{-tx}}{1+x^2}\,dx. \tag{27}$$

What makes this particularly interesting is that if let $t \to 0$, then we immediately recover the value of $\pi/2$ for the integral of $(\sin x)/x$ on $[0,\infty)$. In some sense, the identity (27) generalizes that classic formula. Specifically, it gives us a version that is shifted, or off-set by t. This is most evident when we rewrite (27) as

$$\int_0^\infty \frac{\sin u}{u+t}\,du = \int_0^\infty \frac{e^{-tx}}{1+x^2}\,dx.$$

Some Temporary Comments:

- The identity (27) appears in a comment by Andrey Rekalo writing at the mathematical blog StackExchange.com in 2010.
- The method used to prove (27) can be used to generate other identities, but it is not easy to find any that are quite so nice. Perhaps the simplest alternative to $1/x$ is to take $2/x^2$. The Laplace representation is again easy, and Green's function representation is again easy to write down. Unfortunately it seems mostly unenlightening.
- It would be interesting to give an alternate proof of (27). Contour integration should work, especially after one replaces $\sin u$ with e^{iu} in the shifted formula. The Taylor series method (analysis and synthesis) can also be tried, but one has to deal appropriately with the resulting divergent expansions.

9. Vertical Shift of the x-Axis

One can think of the set $\{x - i\epsilon : x \in \mathbb{R}\}$ as an x-axis that has been shift down by $\epsilon > 0$ in the complex plane. Cauchy's residue calculus then tells us that

$$\int_{-\infty}^\infty \frac{e^{itx}}{x-i\epsilon}\,dt = 2\pi i e^{-\epsilon t}. \tag{28}$$

[Section to be completed ... not working out as I expected.]

10. Inequalities

For the moment, we just hold the places with a list:

$$\left|\frac{\sin x}{x}\right| \leq \min\{1, 1/|x|\} \quad \text{for all } x \in \mathbb{R}.$$

$$\left|\frac{d^n}{dx^n}\frac{\sin x}{x}\right| \leq \frac{1}{n+1} \quad \text{for all } x \in \mathbb{R}.$$

$$1 - \cos 2x \leq 4(1 - \cos x) \quad \text{for all } x \in \mathbb{R}$$

$$0 \leq 3 + 4\cos x + \cos 2x \quad \text{for all } x \in \mathbb{R}$$

11. More ODE Connections

- $J_0(z)$ — Best of the Bessel Functions

  In time, some material on Bessel functions will be added, especially the world's favorite Bessel function $J_0(z)$. For the moment, just ponder the Laplace transform pair:

  $$\int_0^\infty e^{-\lambda t} J_0(t)\, dt = \frac{1}{\sqrt{1+\lambda^2}} \quad \text{and} \quad \int_0^\infty e^{-\lambda t} \sin(t)\, dt = \frac{1}{1+\lambda^2}.$$

- The homogenous ODE

  $$t^2 y''(t) + 2y'(t) + ty(t) = 0$$

  has the independent solutions

  $$y_1(t) = \frac{\sin x}{x} \quad \text{and} \quad y_2(t) = \frac{\cos x}{x}.$$

  This ODE also lives in the web of the Bessel functions, but it is way too early to connect the dots.

12. Use of the Residue Calculus

The essay gets a little ragged at this point. The complex variable material should probably be split off into a separate piece.

Many books on complex variables evaluate the $(\sin x)/x$ integral by the method of residues, but, if this integral is really your only goal, then the residue calculus is not the simplest tool. On the other hand, if you are interested in a wide range of integrals that include the $(\sin x)/x$ integral, the the residue calculus will take your farther and faster than anything else.

The usual approach to the integral of $\sin x/x$ by the residue calculus begins with the function e^{iz}/z. One then needs a relevant integration contour γ, and a sensible place to start is by taking γ to be the contour determined by the segment $\gamma_1 = [\epsilon, R]$, the circular arc γ_2 in the positive half plane from R through iR to R, the segment $\gamma_3 = [-R, -\epsilon]$, and the circular arc γ_4 in the lower half plane from $-\epsilon$ through $-i\epsilon$ to ϵ.

Since the only singularity inside the contour is a simple pole at $z = 0$ and since the residue at $z = 0$ is 1, Cauchy's residue formula tell us the integral over γ is $2\pi i$. Since $\cos(x)/x$ is odd the sum of its integrals over the two segments γ_1 and γ_3 is zero. Moreover, since $\sin(x)/x$ is even, its integrals over γ_1 and γ_3 are equal, so, in summary, one finds

$$\int_{\gamma} \frac{e^{iz}}{z}\, dz = 2\pi i = 2i \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx + \int_{\gamma_2} \frac{e^{iz}}{z}\, dz + \int_{\gamma_4} \frac{e^{iz}}{z}\, dz. \tag{29}$$

The integral over the big arc γ_2 does go to zero as $R \to \infty$, but, since the integrand only decays linearly like $1/R$ and since the path has length $O(R)$, this fact is not trivial. Still, this is not hard to prove, either with help from Jordan's Lemma, or by estimating the integral on the big arc in three pieces — the central part of the arc above the level $H > 0$ and the two subarcs that connect the central part of the arc to the real axis.

Careless thinking might lead one to guess that the integral over the little arc γ_4 is small, but it is not. If we make the substitution $z = \epsilon e^{i\theta}$ then we have

$$\int_{\gamma_4} \frac{e^{iz}}{z}\, dz = \int_{[\pi, 2\pi]} \exp(i\epsilon e^{i\theta}) i\, d\theta \to i\pi \quad \text{as } \epsilon \to 0,$$

so taking the limits $\epsilon \to 0$ and $R \to \infty$ in (29) recaptures our favorite formula.

This derivation may seem complicated in compared to earlier derivations, but this method has the power to go much further. For example, by quite similar calculations one can show

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2} \quad \text{and} \quad \int_0^{\infty} \frac{\sin^3 x}{x^3}\, dx = \frac{3\pi}{2}. \tag{30}$$

The first of these integrals can be found by several methods, but Cauchy's residue calculus seems to be the method of choice for the second integral and its more complicated relatives.

Exercise. Use the residue method to calculate the two integrals of (30). Go further if you like and find the value of

$$I_n = \int_0^{\infty} \frac{\sin^n x}{x^n}\, dx \quad \text{for all } n = 1, 2, \ldots$$

Alternative Contour

The contour we used above is a classic that is applied in many books, but there is a attractive alternative that is perhaps a little more straightforward. In particular, it avoids the need to call on Jordan's Lemma.

The alternative contour is given by the five segments $[\epsilon, R]$, $\gamma_1 = [R, R + iy]$, $\gamma_2 = [R + iy, -R + iy]$, $\gamma_3 = [-R + iy, -R]$, $[-R, -\epsilon]$, and the little arc of radius ϵ that goes under 0 from $-\epsilon$ to ϵ. The calculation goes just as before except we have the simpler estimates

$$\left| \int_{\gamma_1 + \gamma_2 + \gamma_3} \frac{e^{iz}}{z}\, dz \right| \le \frac{y}{R} + (2R)e^{-y} + \frac{y}{R} \le 4ye^{-y/2}$$

where in the last step we took $R = e^{y/2}$ and we assume $y \ge 1$. Now we let $y \to \infty$, and we complete the calculation as before.

This story is a simple reminder that there is always substantial flexibility in the choice of one's contour. Here there was not a major difference, but there was still something to be seen. The big Jordan arc has the charm of being "similar" at all scales, but this symmetry comes at the cost of requiring us to use the specialized Jordan Lemma. The rectangular contour loses some symmetry because of the different rates scaling of the segments, but it has the charm that now one can simply use the plain vanilla "height and length" estimation of the integrals. It is probably a good mental exercise to take almost any contour integral calculation and ask how the computation would change if one were to take a different contour.

13. Sin and Gamma

Surely the most famous connection between the sine and Gamma functions is the *reflection formula*,

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}. \tag{31}$$

This identity is valid for all complex α but for the moment, we just consider real $\alpha \in (0,1)$. The Beta and Gamma functions have the general relation

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

so if we set $\beta = 1-\alpha$, then we see that we can get the reflection formula if we can show

$$\int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\,dx = \frac{\pi}{\sin(\alpha\pi)}.$$

By the change of variables $x = y/(1+y)$, we could equivalently show

$$\int_0^\infty \frac{y^{\alpha-1}}{1+y}\,dy = \frac{\pi}{\sin(\alpha\pi)} \quad \text{for } 0<\alpha<1. \tag{32}$$

This is an integral for which there are many derivations, but if one notices that the integrand has a simple pole at $y=-1$ it is certainly natural to want to give the Residue Theorem a try. Seeing π on the righthand side of this identity offers further encouragement for this approach.

How to Slice It

There are two challenges to using the residue theorem to prove an identity. One needs to find the right contour and one needs to find the right integrand. Here is is perhaps easier to consider the contour first.

The driving consideration is then our need to deal honestly with the fractional power $z^{\alpha-1}$. We can define this as a single-valued function if we restrict our contours to the complex plane with the negative real axis removed. In terms of polar coordinates we consider just $z = re^{i\theta}$ where θ is required to be strictly between $-\pi$ and π.

Now we face some choices depending on how fussy (or careful) we want to be. The least fussy choice is to take the "boundary of the slit disk." This is a fine choice if one is in good practice with contour integration, but most of us may be more comfortable with contours like those given in the figure [Fig to be added.]

We can then contemplate the integral

$$\int_{\gamma(\epsilon,R)} f(z)\,dz = \int_{\gamma(\epsilon,R)} \frac{z^{\alpha-1}}{1-z}\,dz,$$

which the residue theorem tells us is equal to $-2\pi i$ since the only pole inside the contour is at $z=1$, and the residue at $z=1$ is equal to -1. Now we consider the limits when $\epsilon \to 0$ and $R \to \infty$. The integrals over the big circular arc is $O(R^{\alpha-1})$

and the integral over the small circular arc is $O(\epsilon^\alpha)$, so for $0 < \alpha < 1$ both integrals are go to zero in the limit.

We are left to consider the integrals over two successive line segments $L_1(\epsilon, R)$ and $L_2(\epsilon, R)$. If we fix R fixed and let $\epsilon \to 0$ we get

$$\int_{L_1(0,R)} \frac{z^{\alpha-1}}{1-z}\,dz = \int_{-R}^{0} \frac{r^{\alpha-1}e^{i\alpha\pi}}{1+r}\,dr \quad \text{and} \quad \int_{L_2(0,R)} \frac{z^{\alpha-1}}{1-z}\,dz = \int_{0}^{R} \frac{r^{\alpha-1}e^{-i\alpha\pi}}{1+r}\,dr$$

Now, if we add the two integrals and let $R \to \infty$, we find

$$\{-e^{i\alpha} + e^{-i\alpha}\} \int_0^\infty \frac{r^{\alpha-1}}{1+r}\,dr = -2\pi i,$$

from which division gives us the integral (32).

14. Final Scraps and Notes for Further Development

- Comment on Proving Analyticity. In an earlier argument we needed to know that

$$f(z) = \int_0^\infty e^{-xz}\,dx$$

is analytic for $\mathrm{Re} z > 0$.

The proof uses a three step dance. First, one applies the Cauchy integral formula to the analytic integrand e^{-xz}, noting here that for any simple closed smooth path γ in the domain one has

$$\int_\gamma e^{-xz}\,dz = 0.$$

Second, one uses Fubini's theorem to note

$$\int_\gamma f(z)\,dz = \int_0^\infty \int_\gamma e^{-xz}\,dz\,dx = 0.$$

Third, and finally, one uses Morrea's converse of Cauchy's formula. This says that a function that satisfies the Cauchy integral formula on a domain D must be analytic in D, and here we can conclude that $f(z)$ is analytic in $\mathrm{Re} z > 0$.

- More Generic Cases of Integral Representation.

Integral representations are daily fodder in complex variables. Here is a reminder from Bak and Newman (1996), p. 234. To represent the sum

$$f(z) = \sum_{n=0}^{\infty} \frac{n^2}{1+n^2} z^n$$

as an integral, first note that (except for the extra factor of n) we see a familiar Laplace transform

$$\frac{n}{a^2+n^2} = \int_0^\infty e^{-nt} \sin at\,dt.$$

Using the zD operator to give us the extra n, we find after doing the sums and derivative that

$$f(z) = z \int_0^\infty \frac{e^t \cos t}{(e^t - z)^2}\,dt.$$

Among other things, this gives us the analytic continuation of $f(z)$ for the slit domain $\mathcal{C} - [1, \infty)$.

- Special Role of Laplace Representations.

  Every Laplace transform gives one a way to represent a function as an integral. Moreover, in the guts of that integral we have an exponential. This has multiple benefits, including increased access to geometric summation. The most widely used representation is given by the gamma integral,

$$\frac{1}{n^p} = \frac{1}{\Gamma(p)} \int_0^\infty e^{-nt} t^{p-1}\, dt.$$

  Stripping away the clutter, we have just a linear correspondence

$$e^{-nt} \leftrightarrow 1/n^p.$$

  Hence we have the opportunity to apply geometric summation almost anytime we face a sum involving the algebraic terms $1/n^p$. This kind of correspondence has connections to the umbral calculus and to the notion of subordination of stochastic processes.

  Remark. A very attractive way to get the sine integral is to integrate the Dirichlet kernel and to take limits.

- Exercise. A useful test of this principle is to show that Fubini's theorem implies (a) the fundamental theorem of calculus, (b) integration by parts and (c) Taylor's formula (with the integral remainder). You may even find that these derivations hold under slightly more general conditions than the traditional derivations.
- Mellin Connection. For a function $f[0, \infty) \to \mathbb{R}$ the Mellin transform of f is defined by the relation

$$M[f](s) = \int_0^\infty x^{s-1} f(x)\, dx.$$

  Our integral (32) therefore tells us that

$$M[\frac{1}{1+y}](s) = \frac{\pi}{\sin(s\pi)}.$$

- To evaluate the cosine integral we make a new Laplace substitution

$$\frac{1}{t^2} = \int_0^\infty x e^{-tx}\, dx$$

  to get a double integral. This looks very promising since we have the elementary Laplace integral

$$\int (\cos x) e^{-tx}\, dx = \frac{1}{1+t^2} e^{-tx} \left\{ \sin x - t \cos x \right\}$$

  which can be found from the even simpler Laplace integral

$$\int e^{ix} e^{-tx}\, dx = \frac{e^{(i-t)x}}{i-t}.$$

More to come here ...

References

Albano, M., Amdeberhan, T., Beyerstedt, E. and Moll, V. H. (2010), 'The integrals in Gradshteyn and Ryzhik. Part 15: Frullani integrals', *Sci. Ser. A Math. Sci. (N.S.)* **19**, 113–119.